

Expansion and Hidden Dimensions In a New Cosmological Model

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Abstract. In this paper, we present a new cosmological model using fractal manifold. We prove that a space defined by this kind of manifold is an expanding space. This model provides us with consistent arguments pertaining to the relationship between variation of geometry and movement of matter. This study leads to the existence of new fundamental principles. A clear picture is then portrayed about the expansion of the universe represented by fractal manifold.

Key words: Cosmology; Manifold, Fractal Manifold.

MSC Subject Classification: 85A40, 85A04, 58A05.

1 Introduction

The last estimation of the constitution of our universe shows that 73% of the universe is made of *Dark Energy*, and 23% is made of *Dark Matter*, meanwhile the *Normal Matter* constitutes only 4% of the universe. Everything on Earth, everything that we have ever observed with all our instruments constitute 4% of the whole universe, and all our famous theories, laws, formulas revolve around the 4%. Discovering the real shape of the universe may prove unattainable if we take into consideration the fact that more than 95% of the universe is unknown, and that what constitutes the 95% of the universe is huge and so far from us (i.e. billions and billions of light years). This discovery may be even more challenging when we acknowledge the nature of our short life span, human size and illusion which may be caused by our dependency on human observatory tools. Moreover, our observation comes from the reflection of light on observed objects despite the fact that we are not entirely sure of the nature of light or how it crosses the geometry of the universe. The aforementioned statements lead to the questioning of our reliance on analyzes, observations and measures using normal matter generated tools. Normal matter, which constitutes 4% of the universe, can hardly be relied upon to lead to a breakthrough in understanding/discovering the unknown 96%. The Big Bang theory stipulates that normal matter constituted 100% of the whole universe which gives rise to the question regarding the source(s) of Dark energy and Dark matter? Perhaps our misunderstanding of the universe is due to the frequent use of tools and theories adapted to the current normal matter. If this normal matter is positioned and stretched by the last 96%, then perhaps we need to adopt another approach to the problem which is totally different from all existing theories including relativity, gravity and quantum approach. We have to model a mathematical

object which:

- a) fits the apparent nature of the universe,
- b) allows us to talk about expansion and study its different properties (how do space, dimension and matter evolve in the universe with time).

The mathematical object introduced in [1] and called fractal manifold may fit our need since it has the property of an expanding object. It is known that the universe was not static [6, 25, 7], rather it was expanding [9, 10, 14, 15, 16, 17, 18, 12, 8]! This discovery marked the beginning of the modern cosmology [3, 13, 20]. The most fundamental results in modern cosmology are based on observational data and a theoretical model advocated by general relativity. Scientists were surprised by the results¹ of the observation of supernova requiring a shocking change of picture. One of the interpretations of the results indicates that the universe expansion is accelerating. For more than 75 years, including the present day, many scientists have been (and still are) confused about providing a valid/reliable interpretation of the expansion phenomenon [4, 5, 19, 22, 26, 24]. Up until now, we have been able to identify neither the real dimension of our universe, nor the real nature of the expansion, its properties and consequences.

This paper has been able to come up with a direct application of fractal manifold. This application consists of details and analysis related to the expansion of a homogeneous and isotropic space. This study leads to the following outcomes:

- *The universe has geometric properties which are independent of matter that it contains.*
- *The variation of the universe geometry bends the light.*
- *In an expanding space where points are expanding, there is no straight line geodesic. All geodesics are curved due to the expansion of points.*
- *The variation of the universe geometry creates the movement of matter.*
- *The variation of the universe geometry affects the gravity.*

The plan of this work is summarized as follows: In a preliminary part, we present a basic introduction about the mathematical construction of fractal manifolds. In section 3, we present some properties of the expansion in fractal manifold. We prove that the expansion is due to the appearance of new hidden dimensions in section 4. Moreover a consistent analysis about the nature of these hidden dimensions is elaborated (their existence, their order of appearance, their size and number). We establish the Hubble's law in section 5, and we prove that this expansion is bounded. A clear picture is portrayed about the expansion of a universe represented by fractal manifold. Eventually, this study will propose a new scenario and picture of our universe.

2 Preliminary Tools

We introduce basic notions about the fractal manifold model, the reader will find in [1] deep details about the philosophical background relating the construction. Let

¹Two major studies "The supernova cosmology project", and "High-z supernova search team" found evidence for an accelerating universe [11, 21, 23].

f_i , $i = 1, 2, 3$, be three continuous and nowhere differentiable functions, defined on $[a, b] \subset \mathbb{R}$, $a < b$ finite real numbers, where their associated graphs are given by $\Gamma_{i,0}([a, b]) = \left\{ (x, y) \in \mathbb{R}^2 / y = f_i(x), x \in [a, b] \right\}$, $i = 1, 2, 3$. We consider the function $f(x, y) = \frac{1}{2y} \int_{x-y}^{x+y} f(t)dt$, we call forward and backward mean functions of f_i , $i = 1, 2, 3$, the functions given by:

$$f_i(x + \frac{\delta_0}{2}, \frac{\delta_0}{2}) = \frac{1}{\delta_0} \int_x^{x+\delta_0} f_i(t)dt, \quad f_i(x - \frac{\delta_0}{2}, \frac{\delta_0}{2}) = \frac{1}{\delta_0} \int_{x-\delta_0}^x f_i(t)dt, \quad (1)$$

and we denote respectively their associated graphs by: $\Gamma_{i,\delta_0}^\sigma$, $\sigma = \pm$, $i = 1, 2, 3$.

Definition 1 We call small resolution domain, and we denote it by \mathcal{R}_f , the set $\mathcal{R}_f = \{ \delta_0 \in \mathbb{R}^+ / f(x, \delta_0) \text{ is differentiable on } [a, b] \} \cap [0, \alpha]$, where $0 < \alpha \ll 1$, a small real number².

2.1 Local double space

Let us consider $\forall \delta_0 \in \mathcal{R}_f$, the translation $T_{\delta_0} : \prod_{i=1}^3 \Gamma_{i\delta_0}^+ \times \{\delta_0\} \longrightarrow \prod_{i=1}^3 \Gamma_{i\delta_0}^- \times \{\delta_0\}$, defined by: $T_{\delta_0}((a_1, b_1), (a_2, b_2), (a_3, b_3)) = ((a_1 + \delta_0, b_1), (a_2 + \delta_0, b_2), (a_3 + \delta_0, b_3))$, where $(a_i, b_i) \in \Gamma_{i\delta_0}^+$, that is to say $b_i = f_i(a_i + \frac{\delta_0}{2}, \frac{\delta_0}{2}) = \frac{1}{\delta_0} \int_{a_i}^{a_i+\delta_0} f_i(t)dt$, for $i = 1, 2, 3$.

Using this translation we introduce the δ_0 -manifold with a triplet local chart:

Definition 2 Let δ_0 be in \mathcal{R}_f , and M_{δ_0} be an Hausdorff topological space. We say that M_{δ_0} is an δ_0 -manifold if for every point $x \in M_{\delta_0}$, there exist a neighborhood Ω_{δ_0} of x in M_{δ_0} , a map φ_{δ_0} , and two open sets $V_{\delta_0}^+$ of $\prod_{i=1}^3 \Gamma_{i\delta_0}^+ \times \{\delta_0\}$ and $V_{\delta_0}^-$ of $\prod_{i=1}^3 \Gamma_{i\delta_0}^- \times \{\delta_0\}$ such that $\varphi_{\delta_0} : \Omega_{\delta_0} \longrightarrow V_{\delta_0}^+$, and $T_{\delta_0} \circ \varphi_{\delta_0} : \Omega_{\delta_0} \longrightarrow V_{\delta_0}^-$ are two homeomorphisms.

2.2 Fractal manifold: Prototype

In the purpose to define fractal manifold, we will use the notion of diagonal topology introduced in [1]. A diagonal topology $\mathcal{T}_d \subset \mathcal{P}(M)$ of a set $M = \bigcup_{\delta_0 \in \mathcal{R}_f} M_{\delta_0}$ union of Hausdorff topological spaces all disjoint or all the same³, consists of subsets of M that verify the following axioms:

- (i) $\phi \in \mathcal{T}_d$, and $M \in \mathcal{T}_d$,
- (ii) $\omega_1 \in \mathcal{T}_d, \omega_2 \in \mathcal{T}_d \Rightarrow \omega_1 \tilde{\cap} \omega_2 \in \mathcal{T}_d$,
- (iii) $\omega_i \in \mathcal{T}_d, \forall i \in J \Rightarrow \bigcup_{i \in J} \omega_i \in \mathcal{T}_d$,

where $A \tilde{\cap} B = \bigcup_{\delta_0 \in \mathcal{R}_f} (A_{\delta_0} \cap B_{\delta_0})$ for $A = \bigcup_{\delta_0 \in \mathcal{R}_f} A_{\delta_0}$ and $B = \bigcup_{\delta_0 \in \mathcal{R}_f} B_{\delta_0}$ two subsets of M , with $A_{\delta_0}, B_{\delta_0}$ elements of M_{δ_0} for all $\delta_0 \in \mathcal{R}_f$. The elements of \mathcal{T}_d are called open sets, and (M, \mathcal{T}_d) is called diagonal topological space.

²This small real number will be determined later on

³Either $M = \bigcup_{\delta_0 \in \mathcal{R}_f} M_{\delta_0}$ is a disjoint union, or $\forall \delta_0 \in \mathcal{R}_f, M_{\delta_0} = M_0$ and then $M = M_0$.

Let $x : \mathcal{R}_f \longrightarrow M$ be a continuous path on M . If $\forall \delta_0 \in \mathcal{R}_f$, Ω_{δ_0} is an open neighborhood of $x(\delta_0)$ in M_{δ_0} , then the set $\Omega(\text{Range}(x)) = \bigcup_{\delta_0 \in \mathcal{R}_f} \Omega_{\delta_0}$ is called diagonal neighborhood of the set $\text{Range}(x) = \bigcup_{\delta_0 \in \mathcal{R}_f} \{x(\delta_0)\}$ in M .

Definition 3 Let $M = \bigcup_{\delta_0 \in \mathcal{R}_f} M_{\delta_0}$ be an union of Hausdorff topological spaces all disjoint or all the same. We say that M admits an internal structure x on $P \in M$, if there exists a \mathcal{C}^0 parametric path

$$\begin{aligned} x : \mathcal{R}_f &\longrightarrow \bigcup_{\delta_0 \in \mathcal{R}_f} M_{\delta_0} \\ \delta_0 &\longmapsto x(\delta_0) \in M_{\delta_0}, \end{aligned} \quad (2)$$

such that $\forall \delta_0 \in \mathcal{R}_f$, $\text{Range}(x) \cap M_{\delta_0} = \{x(\delta_0)\}$, and there exists $\delta_0' \in \mathcal{R}_f$ such that $P = x(\delta_0') \in M_{\delta_0'}$.

Definition 4 Let $M = \bigcup_{\delta_0 \in \mathcal{R}_f} M_{\delta_0}$ be an union of Hausdorff topological spaces all disjoint or all the same. Let $x : \mathcal{R}_f \subset \mathbb{R} \longrightarrow \bigcup_{\delta_0 \in \mathcal{R}_f} M_{\delta_0}$ be an internal structure on it. We call object of M the set $\text{Range}(x)$.

Definition 5 A diagonal topological space (M, \mathcal{T}_d) is called fractal manifold if $M = \bigcup_{\delta_0 \in \mathcal{R}_f} M_{\delta_0}$, where $\forall \delta_0 \in \mathcal{R}_f$, M_{δ_0} is an δ_0 -manifold, and if $\forall P \in M$, M admits an internal structure x on P such that there exist a neighborhood $\Omega(\text{Range}(x)) = \bigcup_{\delta_0 \in \mathcal{R}_f} \Omega_{\delta_0}$, with Ω_{δ_0} a neighborhood of $x(\delta_0)$ in M_{δ_0} , two open sets $V^+ = \bigcup_{\delta_0 \in \mathcal{R}_f} V_{\delta_0}^+$ and $V^- = \bigcup_{\delta_0 \in \mathcal{R}_f} V_{\delta_0}^-$, where $V_{\delta_0}^\sigma$ is an open set in $\prod_{i=1}^3 \Gamma_{i\delta_0}^\sigma \times \{\delta_0\}$ for $\sigma = \pm$, and there exist two families of maps $(\varphi_{\delta_0})_{\delta_0 \in \mathcal{R}_f}$ and $(T_{\delta_0} \circ \varphi_{\delta_0})_{\delta_0 \in \mathcal{R}_f}$ such that $\varphi_{\delta_0} : \Omega_{\delta_0} \longrightarrow V_{\delta_0}^+$ and $T_{\delta_0} \circ \varphi_{\delta_0} : \Omega_{\delta_0} \longrightarrow V_{\delta_0}^-$ are homeomorphisms for all $\delta_0 \in \mathcal{R}_f$.

Definition 6 A local chart on the fractal manifold M is a triplet $(\Omega, \varphi, T \circ \varphi)$, where $\Omega = \bigcup_{\delta_0 \in \mathcal{R}_f} \Omega_{\delta_0}$ is an open set of M , φ is a family of homeomorphisms φ_{δ_0} from Ω_{δ_0} to an open set $V_{\delta_0}^+$ of $\prod_{i=1}^3 \Gamma_{i\delta_0}^+ \times \{\delta_0\}$, and $T \circ \varphi$ is a family of homeomorphisms $T_{\delta_0} \circ \varphi_{\delta_0}$ from Ω_{δ_0} to an open set $V_{\delta_0}^-$ of $\prod_{i=1}^3 \Gamma_{i\delta_0}^- \times \{\delta_0\}$ for all $\delta_0 \in \mathcal{R}_f$. A collection $(\Omega_i, \varphi_i, (T \circ \varphi)_i)_{i \in J}$ of local charts on the fractal manifold M such that $\bigcup_{i \in J} \Omega_i = \bigcup_{\delta_0 \in \mathcal{R}_f} M_{\delta_0} = M$, where $\bigcup_{i \in J} \Omega_{i, \delta_0} = M_{\delta_0}$, is called an atlas. The coordinates of an object $P \in \Omega$ related to the local chart $(\Omega, \varphi, T \circ \varphi)$ are the coordinates of the object $\varphi(P)$ in $\bigcup_{\delta_0 \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\delta_0}^+ \times \{\delta_0\}$, and of the object $T \circ \varphi(P)$ in $\bigcup_{\delta_0 \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\delta_0}^- \times \{\delta_0\}$.

Lemma 1 Let g_1, g_2 , and g_3 be differentiable functions, and let $g_i(x + \sigma \frac{\delta_1}{2}, \frac{\delta_1}{2})$, be the forward (respectively backward) mean functions of the functions g_i , for $i=1,2,3$. If we associate the graph Γ_{i0} to the functions $g_i(x)$ (respectively, $\Gamma_{i\delta_1}^\sigma$ to the functions $g_i(x + \sigma \frac{\delta_1}{2}, \frac{\delta_1}{2})$, $\sigma = \mp$, for $i=1,2,3$). The product $\prod_{i=1}^3 \Gamma_{i0}$ is a fractal manifold of class \mathcal{C}^1 homeomorphic to $\bigcup_{\delta_1 \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\delta_1}^\sigma \times \{\delta_1\}$.

Theorem 1 If M is a fractal manifold, then $\forall n > 1$, there exist a family of homeomorphisms φ_k , and a family of translations T_k for $2^{n-1} \leq k \leq 2^n - 1$, such that for $\sigma_j = \pm$, $j = 1, \dots, n-2$, one has the 2^{n-1} diagrams given by:

$$\begin{array}{ccc}
M & \xrightarrow{\varphi_k} & \bigcup_{\delta_0 \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \dots \bigcup_{\delta_{n-1} \in \mathcal{R}_{\delta_{n-1}}} \prod_{i=1}^3 \Gamma_{i\delta_{n-1}}^{\sigma_1 \dots \sigma_{n-1}^+} \times \{\delta_{n-1}\} \times \dots \times \{\delta_1\} \times \{\delta_0\} \\
& \searrow T_k \circ \varphi_k & \downarrow T_k \\
& & \bigcup_{\delta_0 \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \dots \bigcup_{\delta_{n-1} \in \mathcal{R}_{\delta_{n-1}}} \prod_{i=1}^3 \Gamma_{i\delta_{n-1}}^{\sigma_1 \dots \sigma_{n-1}^-} \times \{\delta_{n-1}\} \times \dots \times \{\delta_1\} \times \{\delta_0\}
\end{array}$$

Remark 1 To summarize, the definition 2 defines the double homeomorphism at a given scale $\delta_0 \in \mathcal{R}_f$. The definition 5 defines a family of double homeomorphisms for all $\delta_0 \in \mathcal{R}_f$, which gives the step 0, Fig. 1. The lemma 1 induces the existence of 2^2 homeomorphisms which represent the step 1, Fig.1. The definitions 3 and 4 introduce an internal structure that defines an object of the fractal manifold. This internal structure allows an object to obtain a local representative element at every scale. The topology associated to this kind of manifold is a diagonal topology (kind of hausdorff topological space glued together with the internal structure that defines the given object). The theorem 1 induces the existence of 2^n homeomorphisms which represent the step n . All this homeomorphisms exist automatically if we construct the first one that constitutes the step 0. From one step to another we obtain locally appearance of new structures that will be discussed later on in this paper.

Corollary 1 Let g_1, g_2, g_3 be three differentiable functions, and Γ_{i0} be their associated graphs. If M_0 is a three dimensional differentiable manifold homeomorphic to the product $\prod_{i=1}^3 \Gamma_{i0}$, then M_0 is a fractal manifold.

2.3 Elements of fractal manifold

An object P of a fractal manifold M is a set $Range(x)$, where the continuous map $x : \mathcal{R}_f \longrightarrow M$ describes the evolution of one representative element $x(\delta_0) \in M_{\delta_0}$ of x . An element $x(\delta_0)$ of M_{δ_0} is represented in local coordinates by two points, then an object P of M is represented in local coordinates by $Range(x^+) \cup Range(x^-)$ where the paths x^+ and x^- are given by:

$$\begin{array}{ccc}
x^+ : \mathcal{R}_f & \longrightarrow & \bigcup_{\delta_0 \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\delta_0}^+ \times \{\delta_0\} \\
\delta_0 & \longmapsto & \varphi_{\delta_0}(x(\delta_0))
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
x^- : \mathcal{R}_f & \longrightarrow & \bigcup_{\delta_0 \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\delta_0}^- \times \{\delta_0\} \\
\delta_0 & \longmapsto & T_{\delta_0} \circ \varphi_{\delta_0}(x(\delta_0))
\end{array}
\tag{3}$$

In the first step (step 0 Fig.1), an object of a fractal manifold M appears as a disjoint union of sets of points (4) if the used functions $f_i(x)$, $i = 1, 2, 3$ are nowhere differentiable ($0 \notin \mathcal{R}_f$)

$$\bigvee \tag{4}$$

However if the used functions $f_i(x)$, $i = 1, 2, 3$, are differentiable, ($0 \in \mathcal{R}_f$), then an object of a fractal manifold M looks like (5).

$$\bigvee \tag{5}$$

In both case we have $d_h(x^+(\delta_0), x^-(\delta_0)) = \delta_0\sqrt{3}$ for all $\delta_0 \in \mathcal{R}_f$, where d_h is the Hausdorff measure[1]. The lemma 1 means that all classical points⁴ of the object (4) will be transformed into objects (5) via a double homeomorphism, and this procedure is repeated indefinitely following the illustration in Fig.0.

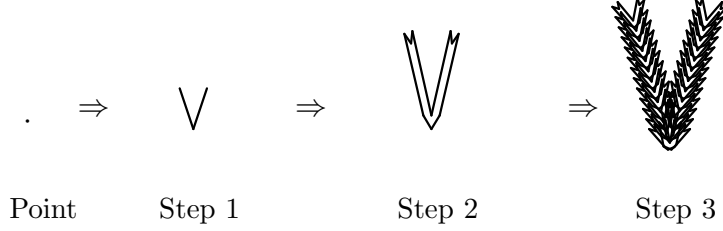


Fig.0 - One illustration of classical point in fractal manifold after 3 steps.

3 Expanding Manifold

The self similarity that appear in the fractal manifold creates a new local structure that leads to the notion of expansion. For better comprehension of this nature we introduce the following:

Definition 7 *We say that a continuous function f is well represented by a family of differentiable functions $(g(x, \delta_0))_{\delta_0}$, $\forall (x, \delta_0) \in [a, b] \times \mathcal{R}_f$ if the function f satisfies in any open neighborhood of $[a, b] \times \mathcal{R}_f$:*

$$f(x) = g(x, \delta_0) + \delta_0 \left(\frac{\partial g(x, \delta_0)}{\partial \delta_0} - \sigma \frac{\partial g(x, \delta_0)}{\partial x} \right), \quad \sigma = \pm. \quad (6)$$

Proposition 1 *Let f be a continuous function on an interval $\mathcal{I} \subset \mathbb{R}$, and $g(x, y)$ be the function given by $g(x, y) = \frac{1}{2y} \int_{x-y}^{x+y} f(t)dt$, then we have:*

\mathcal{P}_1 : *If the function f is nowhere differentiable or differentiable⁵, then f is well represented by the family $\left(g(x + \sigma \frac{\delta_0}{2}, \frac{\delta_0}{2})\right)_{\delta_0}$, for $(x, \delta_0) \in \mathcal{I} \times \mathcal{R}_g$, $\sigma = \pm$.*

Proof: If the function f is nowhere differentiable, \mathcal{P}_1 is a consequence of the Lemma 4, [1] and definition 7. If the function f is differentiable then for $\delta_0 = 0$, the function f is defined by $g(x, 0) = f(x)$, and for $\delta_0 \neq 0$ it is not difficult to prove that for $\sigma = \pm$,

$$f(x) = g(x + \sigma \frac{\delta_0}{2}, \frac{\delta_0}{2}) \left(\frac{\partial g(x + \sigma \frac{\delta_0}{2}, \frac{\delta_0}{2})}{\partial \delta_0} - \sigma \frac{\partial g(x + \sigma \frac{\delta_0}{2}, \frac{\delta_0}{2})}{\partial x} \right).$$

⁴Points that are represented by dots in a classical way.

⁵If the considered function is nowhere differentiable then $0 \notin \mathcal{R}_g$, however $0 \in \mathcal{R}_g$ if it is differentiable.

The different steps of a fractal manifold can be summarized in a diagram given by Fig.1 using $N_{\delta_0 \dots \delta_{j-1}}^{\sigma_1 \dots \sigma_j} = \prod_{i=1}^j \Gamma_{i\delta_{i-1}}^{\sigma_1 \dots \sigma_j} \times \{\delta_{j-1}\} \times \dots \times \{\delta_0\}$, $\sigma_1 = \pm, \dots, \sigma_j = \pm$.

Indeed, in the step 0 of the Fig.1, we have one diagram, which gives locally two disjoint symmetric elements, functions of the scale variable δ_0 . These elements constitute an object of the fractal manifold of dimension 5 (See illustration in (4)). In the step 1, we have appearance of two similar diagrams **2** and **3** and one new scale variable δ_1 as consequence of the property \mathcal{P}_1 . Using the diagram **2** for example (respectively for the diagram **3**), a point is transformed into a new object (See (5)). At this step, the two disjoint elements given by the form (4) are transformed into two new symmetric elements of dimension 5+1. In the step 2, we have appearance of four similar diagrams **4**, **5**, **6** and **7**, as consequence of the property \mathcal{P}_1 , where using the diagram **4** for example, (respectively for the diagram **5**, **6** and **7**), a point is transformed into an object of the form (5). At this step, the two symmetric elements of dimension 5+1 are transformed into two symmetric elements of dimension 5+2, and this procedure can be continued indefinitely. Objects in the Step 0 correspond to the first appearance of the elements of the fractal manifold. Objects in step n, $\forall n \geq 1$, represent the new appearance of the elements of the fractal manifold after n transformations.

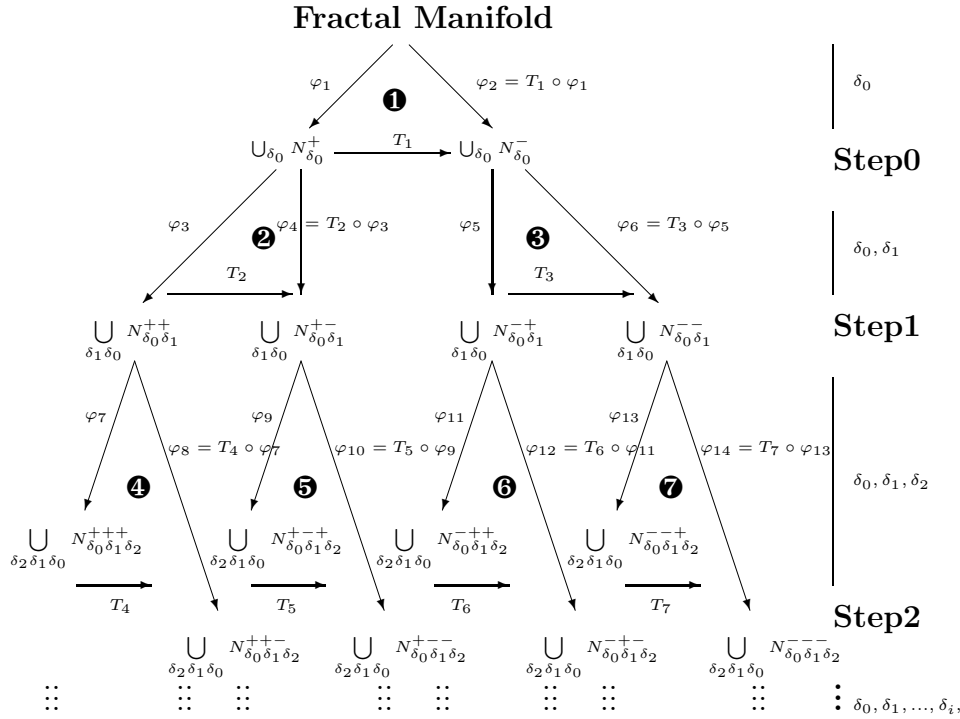


Fig.1. Expanding diagram of a fractal manifold.

3.1 Expanding fractal manifold

Using the diagram (Fig.1), we see that steps in a fractal manifold are not magnification because of the appearance of new structure at every step. This appearance of new structure makes the occupation space bigger. In order to prove that all fractal manifolds are expanding, we define the notion of local expansion. We explain how objects in fractal manifolds are expanding following the different steps of the diagram Fig.1.

Definition 8 *Let M be a fractal manifold, and P be an object of M . We say that the object P is expanding if its local representation at the **step n** is strictly included in its local representation at the **step $n+1$** for all $n \geq 0$.*

Definition 9 *A fractal manifold M is said to be expanding if all object P of M is expanding.*

Using the notations given by the formulas (75),(76),(77),(78) of $y^{\sigma_1\sigma_2}$, $\sigma_1 = \pm$, $\sigma_2 = \pm$ introduced in Appendix A, and using the notation given by the formulas (71), (72) of y^σ for $\sigma = \pm$ (introduced in Appendix A), we obtain the following:

Lemma 2 *For $i = 1, 2, 3$, we have*

$$y_i^{++}(x_i, \delta_0, 0) = y_i^{+-}(x_i, \delta_0, 0) = y_i^+(x_i, \delta_0), \quad \forall \delta_0 \in \mathcal{R}_f$$

$$y_i^{-+}(x_i, \delta_0, 0) = y_i^{--}(x_i, \delta_0, 0) = y_i^-(x_i, \delta_0), \quad \forall \delta_0 \in \mathcal{R}_f$$

Proof: One can find the result using $y_i^{-+}(x_i, \delta_0, 0) = \lim_{\delta_1 \rightarrow 0} y_i^{-+}(x_i, \delta_0, \delta_1)$.

Using the previous lemma and notation of Appendix A, we obtain

Theorem 2 *The local representation of object in fractal manifold verifies:*

$$\left(Rg(x^+) \cup Rg(x^-) \right) \subset \left(Rg(x^{++}) \cup Rg(x^{+-}) \cup Rg(x^{-+}) \cup Rg(x^{--}) \right) \quad (7)$$

Proof: See Appendix A.

Remark 2 *The last theorem means that the local representation of an object P in the fractal manifold is expanding from step 0 to step 1. A more general result can be elaborated for all $n \geq 0$ by induction over n , and using the proof of the theorem 2.*

Theorem 3 *All fractal manifolds are expanding manifolds.*

Proof: Using Theorem 2 and by induction over steps, one can find the result.

4 Expansion Parameters and Hidden Dimensions

The expansion of the fractal manifold is characterized by the appearance of new dimensions (new variables δ_i , Fig.1). What about these dimensions? Are they finite or not? Are they small or big? Is this expansion infinite or finite? What kind of relationship have we between the new dimensions and the classical space-time dimensions (x, y, z, t) ? Following the diagram given in Fig.1, we have appearance of new dimensions for every step that induces an appearance of new structure, and then creates the expansion of the space. For more simplicity, we will consider the case where our manifold fits the cosmological principle⁶.

4.1 Properties and expansion parameters

Definition 10 *Let M be fractal manifold, we say that M is homogeneous if all objects of M have same size at a given step.*

Proposition 2 *Every object of a fractal manifold M is expanding symmetrically.*

Proof: One object of the fractal manifold is locally composed by two symmetric strings of length L . Because of the translation T_1 between $\bigcup_{\delta_0} N_{\delta_0}^+$ and $\bigcup_{\delta_0} N_{\delta_0}^-$ the two strings are copies one of the other (Fig.1). In the step 1, the two strings are expanding (Theorem 1) symmetrically. Every string is expanding symmetrically because of the translation T_2 between $\bigcup_{\delta_1 \delta_0} N_{\delta_0 \delta_1}^{++}$ and $\bigcup_{\delta_1 \delta_0} N_{\delta_0 \delta_1}^{+-}$, and the translation T_3 between $\bigcup_{\delta_1 \delta_0} N_{\delta_0 \delta_1}^{-+}$ and $\bigcup_{\delta_1 \delta_0} N_{\delta_0 \delta_1}^{--}$. By induction over n , and using steps, the symmetry can be justified because of the translation T_{2n} and T_{2n+1} , $\forall n > 1$.

Proposition 3 *In an homogeneous fractal manifold M , the distance between two objects is proportional to their initial distance after one step.*

Proof: To study the expansion in a fractal manifold, it is sufficient to study the expansion of a classical point to an object given by (5) (because of this transformation, our space expands in each step). Let us consider a closed and bounded interval I of length $l_0 > 0$. We put one ball at the extremity $\inf I$ and another ball at the extremity $\sup I$, such that the distance between the two balls is l_0 . Let us consider a subdivision of I of length $d > 0$

$$\inf I = \beta_0 < \beta_1 < \beta_2 < \dots < \beta_k = \sup I, \quad (8)$$

and let $I_j = [\beta_j, \beta_{j+1}]$, $j = 0, 1, \dots, k-1$. The family of sets I_j , $j = 0, 1, \dots, k-1$ constitutes a finite covering family of I . Let us consider a family of circles \mathcal{C}_j , such that

$$\mathcal{C}_j \cap I_j = \{\beta_j, \beta_{j+1}\}, \quad j = 0, 1, \dots, k-1. \quad (9)$$

⁶Cosmological Principle: The fractal manifold is homogeneous and isotropic.

The diameter of \mathcal{C}_j is equal to d and

$$l_0 = kd. \quad (10)$$

After one step, δ_1 appears as a new variable, and the family of classical circles \mathcal{C}_j will be transformed in new family of circles that will be called scale circles given by Fig.2. Every new scale circle is circumscribed in a sphere \mathcal{S}_j , $j = 0, 1, \dots, k-1$, of radius R (the expansion increases the dimension). Because of the homogeneity, the new length of our interval I becomes $L = 2Rk$ after one expansion. To calculate this new length L , we have to calculate the radius of the sphere \mathcal{S}_j for $j \in \{0, 1, \dots, k-1\}$.

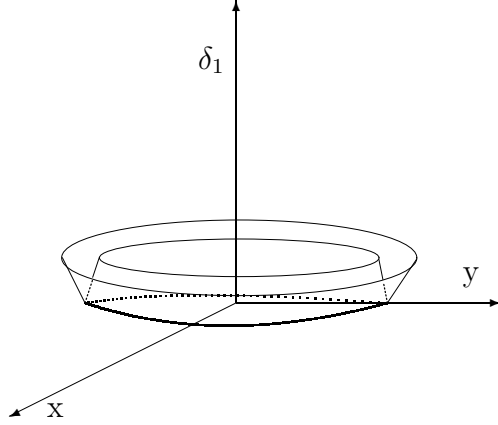


Fig.2. Scale circle

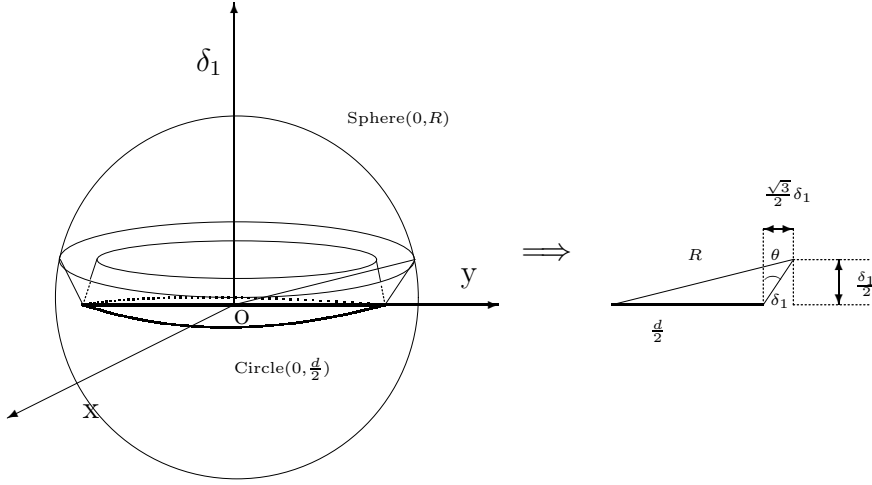


Fig.3. Scale circle in regular fractal manifold

Following Fig.3, we have

$$R^2 = \left(\frac{d}{2} + \frac{\sqrt{3}}{2}\delta_1\right)^2 + \left(\frac{\delta_1}{2}\right)^2, \quad \text{which gives} \quad R = \frac{1}{2}\sqrt{(d + \sqrt{3}\delta_1)^2 + (\delta_1)^2},$$

then $2R = \sqrt{(d + \sqrt{3}\delta_1)^2 + (\delta_1)^2}$, and $2Rk = k\sqrt{(d + \sqrt{3}\delta_1)^2 + (\delta_1)^2}$ which gives $L = \sqrt{(l_0 + k\sqrt{3}\delta_1)^2 + (k\delta_1)^2}$ and using (10), we obtain $L = \sqrt{(l_0 + \frac{l_0}{d}\sqrt{3}\delta_1)^2 + (\frac{l_0}{d}\delta_1)^2}$, which gives the proportionality law function of the initial distance l_0

$$L = l_0 \sqrt{1 + 2\frac{\sqrt{3}\delta_1}{d} + 4\left(\frac{\delta_1}{d}\right)^2} = l_0 a_1, \quad (11)$$

where $a_1 = \sqrt{1 + 2\frac{\sqrt{3}\delta_1}{d} + 4\left(\frac{\delta_1}{d}\right)^2}$.

Proposition 4 *In an homogeneous fractal manifold M , the so far the centers are between objects, the bigger distance they have between them after steps.*

Proof: Indeed, using formula (11), the recession $L - l_0$ between balls after one expansion is given by

$$L - l_0 = l_0 \left(\sqrt{1 + 2\frac{\sqrt{3}\delta_1}{d} + 4\left(\frac{\delta_1}{d}\right)^2} - 1 \right). \quad (12)$$

From this formula we deduce that the greater distance we have between balls, the bigger recession we have after expansion.

Remark 3 *The centers of any two distant objects in a fractal manifold M appear moving away one from the other after every step.*

Proposition 5 *Let M be an homogeneous fractal manifold, A and B be two distant fixed objects on it. If the distance between A and B is equal to $l_0 > 0$, and equal to L_n after n steps, then we have the following proportionality law:*

$$L_n = l_0 \prod_{i=1}^n a_i, \quad (13)$$

where $a_i = \sqrt{1 + 2\sqrt{3}\frac{\delta_i}{d} + 4\left(\frac{\delta_i}{d}\right)^2}$ is called the i -th expansion parameter, δ_i is a small real number $\forall i = 1, \dots, n$.

Proof: From the formula (11) we have

$$L_1 = l_0 \sqrt{1 + 2\sqrt{3}\frac{\delta_1}{d} + 4\left(\frac{\delta_1}{d}\right)^2}. \quad (14)$$

The distance L_1 is now considered as an initial distance, and if we repeat the procedure of expansion for the step 2 (for which we have appearance of the new variable δ_2). Using the proof of the Properties 3, we find

$$L_2 = L_1 \sqrt{1 + 2\sqrt{3}\frac{\delta_2}{d} + 4(\frac{\delta_2}{d})^2}, \quad (15)$$

where δ_2 is a new real number. By induction over n , one can find the result.

Remark 4 1) Every new variable δ_i is independent from the other variables and then constitutes a new dimension (see [1], theorem 1).

2) The quantity

$$A_n = \prod_{i=1}^n a_i = \prod_{i=1}^n \sqrt{1 + 2\sqrt{3}\frac{\delta_i}{d} + 4(\frac{\delta_i}{d})^2}, \quad (16)$$

is called the n -th step partial expansion parameter.

Corollary 2 Let M be an homogeneous fractal manifold, A and B be two distant objects on it. If the distance between A and B after n steps is equal to L_n , then we have:

$$L_n = a_n L_{n-1}, \quad \text{with} \quad a_n = \sqrt{1 + 2\sqrt{3}\frac{\delta_n}{d} + 4(\frac{\delta_n}{d})^2}, \quad (17)$$

where δ_n is a real number $\forall n \geq 1$.

Proof: Using the formula (14) and by induction over n , we can find the result.

4.2 The hidden dimensions

To end up with a detailed description of the new variables that appear after steps, we need to elaborate three parts:

- i) Introduction of the local small resolution domain.
- ii) Introduction of the common local small resolution domain.
- iii) Determination of the countable family of variables δ_i that appear after steps.

4.2.1 The local small resolution domain

In the small resolution domain \mathcal{R}_f introduced in the construction of fractal manifold (Definition 1), there is no control of the real number α , except it is small. Since we built a fractal manifold using the graphs of mean functions of a given continuous and nowhere differentiable function f , it is then natural to incorporate some conditions on the real number α that reflect the nature of the function f used in this construction.

Definition 11 Let f be a function defined on a given interval $[a, b]$. If for all $x_0 \in]a, b[$, and for all $\theta > 0$, there exists $\delta_0(\theta, x_0) > 0$ such that for $|x - x_0| \leq \delta_0$, we have $|f(x) - f(x_0)| \leq \theta$, then we define the local small resolution domain of the function f at the point x_0 and we denoted by $l\mathcal{R}_{f(x_0)}$ the set

$$l\mathcal{R}_{f(x_0)} = \{ \varepsilon \in \mathbb{R}^+ / f(x_0, \varepsilon) \text{ is differentiable on }]x_0 - \delta_0, x_0 + \delta_0[\} \cap [0, \delta_0] \quad (18)$$

4.2.2 The common local small resolution domain

Using the local small resolution domain, we introduced a local information in relation with the continuity of the functions used in the construction of the fractal manifold. As a matter of fact, the local small resolution domain's definition gives us

$$l\mathcal{R}_{f_1(x_0)} \neq l\mathcal{R}_{f_2(x_0)} \neq l\mathcal{R}_{f_3(x_0)}, \quad \text{whereas} \quad \mathcal{R}_{f_1} = \mathcal{R}_{f_2} = \mathcal{R}_{f_3}. \quad (19)$$

Since we use three different functions in the construction of fractal manifold, and to use the local small resolution domain's definition, we need to introduce the following definition:

Definition 12 *Let us consider three different continuous functions f_1 , f_2 , and f_3 defined on $[a, b]$, where the associated local small resolution domain are $l\mathcal{R}_{f_1(x)} \neq l\mathcal{R}_{f_2(x)} \neq l\mathcal{R}_{f_3(x)}$, $x \in]a, b[$. We call common local small resolution domain associated to the functions f_1 , f_2 , and f_3 , the set*

$$\mathfrak{R}_f = l\mathcal{R}_{f_1(x_0)} \cap l\mathcal{R}_{f_2(x_0)} \cap l\mathcal{R}_{f_3(x_0)} \quad (20)$$

Substituting \mathfrak{R}_f for \mathcal{R}_f within definition 5 allows us to use the local small resolution domain in the definition of fractal manifold.

4.2.3 About hidden dimensions

Because of the appearance of new variables from nowhere, we call them "hidden dimensions", and we have:

Theorem 4 *If M is a fractal manifold constructed via the graphs of continuous and nowhere differentiable functions f_i , $i = 1, 2, 3$, then there exist an infinity of real numbers δ_j , for all $j \geq 0$, called hidden dimensions, such that:*

- i) *For all $j \geq 0$, $\delta_{j+1} < \delta_j$.*
- ii) *The n -th partial sum $S_n = \sum_{j=0}^n \delta_j$ converges.*
- iii) *For all $j \geq 0$, $[0, \delta_{j+1}] \subset [0, \delta_j]$.*

Proof: to prove this theorem we proceed as follow. Firstly, we prove by induction the existence of n decreasing variables δ_{ni} , $i = 1, 2, 3$, $n \geq 1$. Secondly, we use the common local small resolution domain to define the variables δ_j , $\forall j \geq 0$, and to conclude. Let f_i be a continuous and nowhere differentiable function in a nonempty interval $[a, b] \subset \mathbb{R}$, for $i = 1, 2, 3$.

1) Mean of f_i : The means of f_i in $[x, x + \delta_{1i}]$, with $x \in]a, b[$, are given by $f_i(x + \sigma \frac{\delta_{1i}}{2}, \frac{\delta_{1i}}{2})$, for $\sigma = \pm$, defined by the formula (1). For more simplicity, we will consider only the case where $\sigma = +$, and we denote $f_{\delta_{1i}}(x) = f_i(x + \frac{\delta_{1i}}{2}, \frac{\delta_{1i}}{2})$, the same proof can be done for $\sigma = -$. If f_i is continuous, then we have $\lim_{\delta_{1i} \rightarrow 0} f_{\delta_{1i}}(x) = f_i(x)$. Indeed, $\forall \varepsilon_i > 0$, there exists $\delta_{0i}(\varepsilon_i, x) > 0$ such that for $|t - x| \leq \delta_{0i}$, $|f_i(t) - f_i(x)| \leq \varepsilon_i$. We deduce by the mean theorem that for any $\delta_{1i} < \delta_{0i}$, we have

$$\int_x^{x+\delta_{1i}} (f_i(t) - f_i(x)) dt \leq \varepsilon_i \delta_{1i}, \quad (21)$$

which gives $|f_{\delta_{1i}}(x) - f_i(x)| \leq \varepsilon_i$, and then $]0, \delta_{0i}] = l\mathcal{R}_{f_i(x)}$ is the local small resolution domain for the function $f_i(t)$ (the value δ_{0i} is function of (ε_i, x) , we can obtain $\delta_{0i}(\varepsilon_i)$ independent of x if we use the uniform continuity of the functions f_i in the interval $[x, x + \delta_{1i}]$, for $i = 1, 2, 3$).

2) Mean of the mean of f_i : The mean of $f_{\delta_{1i}}$ in $[x, x + \delta_{2i}]$ is defined by

$$f_{\delta_{1i}\delta_{2i}} = \frac{1}{\delta_{2i}} \int_x^{x+\delta_{2i}} f_{\delta_{1i}}(t) dt$$

Since $f_{\delta_{1i}}(x)$ is continuous, then we have $\lim_{\delta_{2i} \rightarrow 0} f_{\delta_{1i}, \delta_{2i}}(x) = f_{\delta_{1i}}(x)$, for $\delta_{2i} < \delta_{1i} < \delta_{0i}$. Indeed, $\forall \varepsilon'_i > 0$, there exists $\lambda'_{0i} = \delta_{1i}$ such that for $|t - x| \leq \delta_{1i}$, we have $|f_{\delta_{1i}}(t) - f_{\delta_{1i}}(x)| \leq \varepsilon'_i$. To prove the existence of λ'_0 , it is sufficient to see that $|f_{\delta_{1i}}(t) - f_{\delta_{1i}}(x)| = |f_{\delta_{1i}}(t) - f_i(t) + f_i(t) - f_i(x) + f_i(x) - f_{\delta_{1i}}(x)| \leq |f_{\delta_{1i}}(t) - f_i(t)| + |f_i(t) - f_i(x)| + |f_i(x) - f_{\delta_{1i}}(x)| \leq 3\varepsilon_i$ and then for $\varepsilon'_i = 3\varepsilon_i$, there exists $\lambda'_{0i} = \delta_{1i}$ such that for $|t - x| \leq \delta_{1i}$, we have $|f_{\delta_{1i}}(t) - f_{\delta_{1i}}(x)| \leq \varepsilon'_i$. We deduce by the mean theorem that for any $\delta_{2i} < \delta_{1i}$, we have

$$\int_x^{x+\delta_{2i}} (f_{\delta_{1i}}(t) - f_{\delta_{1i}}(x)) dt \leq \varepsilon'_i \delta_{1i}, \quad (22)$$

which gives $|f_{\delta_{1i}\delta_{2i}}(x) - f_{\delta_{1i}}(x)| \leq \varepsilon'_i$.

The function $f_{\delta_{1i}\delta_{2i}}(x) = \frac{1}{\delta_{2i}} \frac{1}{\delta_{1i}} \int_x^{x+\delta_{2i}} \int_t^{t+\delta_{1i}} f(s) ds dt$ is well defined for $|t - x| \leq \delta_{2i}$ and $|s - t| \leq \delta_{1i}$ which gives $|s - x| \leq \delta_{2i} + \delta_{1i}$, and the interval $[0, \delta_{1i}] = l\mathcal{R}_{f_{\delta_{1i}}(t)}$ constitutes the local small resolution domain for the differentiable function $f_{\delta_{1i}}(t)$, for $i = 1, 2, 3$.

3) By induction over n , and using the continuity of $f_{\delta_{1i} \dots \delta_{(n-1)i}}$ in $[x, x + \delta_{ni}]$, with the condition given by $\lim_{\delta_{ni} \rightarrow 0} f_{\delta_{1i} \dots \delta_{(n-1)i} \delta_{ni}}(x) = f_{\delta_{1i} \dots \delta_{(n-1)i}}(x)$, it is not difficult to see that $\forall \varepsilon_i > 0$, there exists $\delta_{ni} > 0$, such that for $|x - t| < \delta_{ni}$, we have

$$\left| \int_x^{x+\delta_{ni}} f_{\delta_{1i} \dots \delta_{(n-1)i}}(t) dt - f_{\delta_{1i} \dots \delta_{(n-1)i}}(x) \right| \leq \varepsilon_i \delta_{ni}. \quad (23)$$

To find the formula (23), we use the mean theorem in $[x, x + \delta_{ni}]$ and then we obtain the following condition

$$\delta_{ni} < \delta_{(n-1)i} < \dots < \delta_{1i} < \delta_{0i} \quad i = 1, 2, 3. \quad (24)$$

The function

$$f_{\delta_{1i} \dots \delta_{ni}}(x) = \frac{1}{\delta_{ni} \dots \delta_{1i}} \int_x^{x+\delta_{ni}} \dots \int_{t_1}^{t_1+\delta_{1i}} f_i(t) dt dt_1 \dots dt_{n-1} \quad (25)$$

is defined for $|t_{n-1} - x| \leq \delta_{ni}$, $|t_{n-2} - t_{n-1}| \leq \delta_{(n-1)i}$, ..., $|t_1 - t_2| \leq \delta_{2i}$, $|t - t_1| \leq \delta_{1i}$ which gives $t \leq x + \sum_{j=1}^n \delta_{ji}$. The sum $\sum_{j=1}^n \delta_{ji}$, $i = 1, 2, 3$, must be finite as n approaches $+\infty$, because the function f_i are defined and continuous on $[a, b] \subset \mathbb{R}$, that is to say the function $f_{\delta_{ni} \dots \delta_{1i}}$ is well defined as n approaches $+\infty$ for a convergent series $\sum_{j=1}^n \delta_{ji}$, $i = 1, 2, 3$, and we have $[0, \delta_{(j+1)i}] \subset [0, \delta_{ji}] \subset [0, \delta_{0i}]$ for all $j \geq 1$, $i = 1, 2, 3$. The

interval $[0, \delta_{ni}] = l\mathcal{R}_{f_{\delta_{1i} \dots \delta_{(n-1)i}}(t)}$ constitutes the local small resolution domain for the differentiable function $f_{\delta_{1i} \dots \delta_{(n-1)i}}(t)$.

To finalize the proof of this theorem, we need the following lemma:

Lemma 3 *Let δ_{ni} be real numbers for all $n \geq 0$, and $i = 1, 2, 3$.*

If $\delta_{ni} < \delta_{(n-1)i}$, $\forall n > 0$, $i = 1, 2, 3$, then $\min_{i \in \{1,2,3\}} \delta_{ni} < \min_{i \in \{1,2,3\}} \delta_{(n-1)i}$, $\forall n > 0$.

Proof: Suppose that $\delta_{ni} < \delta_{(n-1)i}$, $\forall n > 0$, $i = 1, 2, 3$, and there exists $n_0 > 0$ such that $\min_{i \in \{1,2,3\}} \delta_{n_0 i} \geq \min_{i \in \{1,2,3\}} \delta_{(n_0-1)i}$. Let's suppose for example that $\min_{i \in \{1,2,3\}} \delta_{n_0 i} = \delta_{n_0 1}$ and $\min_{i \in \{1,2,3\}} \delta_{(n_0-1)i} = \delta_{(n_0-1)2}$, then we have $\delta_{n_0 2} \geq \delta_{(n_0-1)2}$, which is impossible, and we have the result.

Following the beginning of the proof of the theorem, we find out that for the three continuous functions f_i , $i = 1, 2, 3$, we have for all $n > 0$

$$\delta_{n1} < \delta_{(n-1)1} < \dots < \delta_{11} < \delta_{01} \quad \text{for } f_1 \quad (26)$$

$$\delta_{n2} < \delta_{(n-1)2} < \dots < \delta_{12} < \delta_{02} \quad \text{for } f_2 \quad (27)$$

$$\delta_{n3} < \delta_{(n-1)3} < \dots < \delta_{13} < \delta_{03} \quad \text{for } f_3. \quad (28)$$

To construct a fractal manifold and to obtain a full diagram⁷, we have to determine following sets:

The common local small resolution domain for the functions f_i , $i = 1, 2, 3$:

$$\mathfrak{R}_f = l\mathcal{R}_{f_1(x)} \cap l\mathcal{R}_{f_2(x)} \cap l\mathcal{R}_{f_3(x)} =]0, \min_{i \in \{1,2,3\}} \delta_{0i}], \quad (29)$$

that will be denoted by \mathfrak{R}_{δ_0} .

The common local small resolution domain for the functions $f_{\delta_{1i}}$, $i = 1, 2, 3$:

$$\mathfrak{R}_{\delta_1} = l\mathcal{R}_{f_{\delta_{11}}(x)} \cap l\mathcal{R}_{f_{\delta_{12}}(x)} \cap l\mathcal{R}_{f_{\delta_{13}}(x)} = [0, \min_{i \in \{1,2,3\}} \delta_{1i}]. \quad (30)$$

The common local small resolution domain for the functions $f_{\delta_{1i} \dots \delta_{ni}}$, $i = 1, 2, 3$, $\forall n \geq 1$:

$$\mathfrak{R}_{\delta_n} = l\mathcal{R}_{f_{\delta_{11} \dots \delta_{n1}}(x)} \cap l\mathcal{R}_{f_{\delta_{12} \dots \delta_{n2}}(x)} \cap l\mathcal{R}_{f_{\delta_{13} \dots \delta_{n3}}(x)} = [0, \min_{i \in \{1,2,3\}} \delta_{ni}]. \quad (31)$$

Finally, by substituting \mathfrak{R}_{δ_0} for the small resolution domain of the functions f_i in Definition 5 (substitute δ_0 for ε), and using \mathfrak{R}_{δ_i} , $i \geq 1$ in the Theorem 1, we can confirm that there exist an infinity of real numbers $\delta_j = \min_{i \in \{1,2,3\}} \delta_{ji}$, we have automatically the n-th partial sum $S_n = \sum_{j=0}^n \delta_j$ which converges, and for all $j \geq 0$, $[0, \delta_{j+1}] \subset [0, \delta_j]$.

Corollary 3 *With the previous notations we have*

$$\forall n > 1, \quad \mathfrak{R}_{\delta_n} \subset \dots \subset \mathfrak{R}_{\delta_2} \subset \mathfrak{R}_{\delta_1}. \quad (32)$$

$$\forall i \geq 1, \quad \mathfrak{R}_{\delta_0} \cap \mathfrak{R}_{\delta_i} =]0, \delta_i]. \quad (33)$$

⁷Definition 5, and Theorem 1, Fig.1.

Remark 5 *The dimension of a fractal manifold corresponds to the number of all independent parameters used in the model: the classical variables (x,y,z,t) plus an infinite number of hidden dimensions δ_i . These last variables were hidden because of their size and appear with an order that follows their size. Their number is infinite and they vary in the given small compact sets \mathfrak{R}_{δ_i} , $\forall i \geq 0$. These compact sets are nested. The extra dimensions can be understood via the family of homeomorphisms constructed in the fractal manifold, indeed, following the diagram given by Theorem 1, these dimensions depend on the step of the expansion. In that space, the three classical spacial dimensions have macroscopic size, that is why we perceive them, and the time varies in a closed set, meanwhile, the additional dimensions are not perceived because of their size.*

The following propositions come directly from the natural construction of the fractal manifold M :

Proposition 6 *The expansion of the homogeneous fractal manifold M creates the motion of the centers of its objects.*

Proposition 7 *The appearance of the hidden dimensions on an homogeneous fractal manifold is due to the existence of constant internal structures into it.*

Proposition 8 *The expansion of the homogeneous fractal manifold M is due to the appearance of the hidden dimensions.*

Proposition 9 *The fractal manifold is expanding in dimensions for every step.*

Proposition 10 *At the step n , the dimension of the fractal manifold is equal to $n+5$ for all $n \geq 0$.*

4.3 A bounded expansion

With the details obtained about the hidden dimensions, we are able now to determine if the expansion is bounded or not.

Theorem 5 *Let M be an homogeneous fractal manifold, A and B be two distant fixed objects on it. If the distance between A and B is equal to $l_0 > 0$, and equal to L_n after n steps, then distance L_n is increasing and bounded.*

Proof: Using formula (13), we have:

1) $L_n < L_{n+1}$, $\forall n \geq 0$.

2) The product $\prod_{i=1}^n \sqrt{1 + 2\frac{\sqrt{3}\delta_i}{d} + 4(\frac{\delta_i}{d})^2}$ converges as n tends to $+\infty$.

Indeed, we have

$$\prod_{i=1}^n \sqrt{1 + 2\frac{\sqrt{3}\delta_i}{d} + 4(\frac{\delta_i}{d})^2} = \exp \left(\ln \left(\prod_{i=1}^n \sqrt{1 + 2\frac{\sqrt{3}\delta_i}{d} + 4(\frac{\delta_i}{d})^2} \right) \right)$$

$$= \exp \left(\sum_{i=1}^n \ln \left(\sqrt{1 + 2\frac{\sqrt{3}\delta_i}{d} + 4\left(\frac{\delta_i}{d}\right)^2} \right) \right) = \exp \left(\frac{1}{2} \sum_{i=1}^n \ln \left(1 + 2\frac{\sqrt{3}\delta_i}{d} + 4\left(\frac{\delta_i}{d}\right)^2 \right) \right),$$

and then $\ln \left(1 + 2\frac{\sqrt{3}\delta_i}{d} + 4\left(\frac{\delta_i}{d}\right)^2 \right) \approx \frac{2\sqrt{3}\delta_i}{d} + 4\left(\frac{\delta_i}{d}\right)^2$, as n approaches infinity.

The convergence of $\sum_{i=1}^n \delta_i$ guaranties the convergence of

$$\sum_{i=1}^n \ln \left(1 + 2\frac{\sqrt{3}\delta_i}{d} + 4\left(\frac{\delta_i}{d}\right)^2 \right). \quad (34)$$

3) Because of the convergence of (34), we have

$$\exp \left(\frac{1}{2} \sum_{i=1}^n \ln \left(1 + 2\frac{\sqrt{3}\delta_i}{d} + 4\left(\frac{\delta_i}{d}\right)^2 \right) \right) \leq C e^l$$

where $l = \lim_{n \rightarrow +\infty} \sum_{i=1}^n \ln \left(1 + 2\frac{\sqrt{3}\delta_i}{d} + 4\left(\frac{\delta_i}{d}\right)^2 \right)$, which confirms the result.

Corollary 4 *All fractal manifold has a bounded expansion.*

5 Hubble's Law in an Homogeneous Space

We know that an universe defined by fractal manifold is increasing and bounded. In the purpose to study the nature of this kind of expansion, we need to involve time in our construction. Until now, we found out the hidden dimensions, and we can determine their sizes, but there is no information about how these dimensions evolve with time to reach their maximum size. This information is lost because of the uniform convergence of the mean functions used in the construction.

5.1 Evolution of dimensions with time

In the following, we introduce a new sequence of continuous functions that allows us to involve time in the hidden dimensions to study a continuous expansion of fractal manifold.

Definition 13 *Let $\delta_i, \forall i \geq 0$ be the hidden dimensions. We call hidden variables a sequence of continuous functions $(\lambda_i(t))_{i \geq 0}$ given by:*

$$\begin{cases} \lambda_i : \mathbb{R}^+ \longrightarrow [0, \delta_i[\\ t \longmapsto \lambda_i(t) \end{cases} \quad (35)$$

such that

- i) $\forall i \geq 0$, the function $\lambda_i(t)$ is increasing.
- ii) $\forall i \geq 0$, $\lim_{t \rightarrow +\infty} \lambda_i(t) = \delta_i$.
- iii) $\forall i \geq 0$, $\forall t \in \mathbb{R}^+$, $\lambda_{i+1}(t) < \lambda_i(t)$.
- iv) $\sum_{i=0}^n \lambda'_i(t)$ converges uniformly.

In the proof of Theorem 4, we have seen that using the mean theorem, we found the formula (21) for any $\delta_{1i} < \delta_{0i}$, the formula (22) for any $\delta_{2i} < \delta_{1i} < \delta_{0i}$, and the formula (23) for any $\delta_{ni} < \delta_{(n-1)i} < \dots < \delta_{2i} < \delta_{1i} < \delta_{0i}$. The uniform convergence makes that $\delta_{ni}, \delta_{(n-1)i}, \dots, \delta_{2i}, \delta_{1i}, \delta_{0i}$ are not functions of t . In the objective to make them functions of time, we use the hidden variables $\lambda_i(t)$ introduced in definition 13 (which verify all properties given by theorem 4), the mean theorem, and the proof of theorem 4 to obtain for all $i \geq 0$

$$\left| \int_x^{x+\lambda_i(t)} f_{\lambda_1 \dots \lambda_{n-1}}(t) dt - f_{\lambda_1 \dots \lambda_{i-1}}(x) \right| \leq \varepsilon \lambda_i(t), \quad (36)$$

and the set of functions that verify the properties of definition 13 is not empty:

Example 1 *An example of hidden variables is given by*

$$\begin{cases} \lambda_n : [0, +\infty[\longrightarrow [0, \delta_n[\\ t \longmapsto \lambda_n(t) = \delta_n - \delta_n e^{-t^2}. \end{cases} \quad (37)$$

where δ_n for all $n \geq 0$ are the hidden dimensions.

5.2 Hubble's law for the first step

To study the nature of the expansion of a fractal manifold, we replace in the formula (17) the hidden dimensions $\delta_i, \forall i \geq 1$, by the hidden variables $\lambda_i(t), \forall i \geq 1, \forall t \in \mathbb{R}^+$, to obtain

$$l_1(t) = l_0 \sqrt{1 + \frac{2\sqrt{3}\lambda_1(t)}{d} + 4\frac{\lambda_1^2(t)}{d^2}}, \quad (38)$$

which yields

$$l_1(t) = l_0 a_1(t), \quad (39)$$

with $a_1(t) = \sqrt{1 + \frac{2\sqrt{3}\lambda_1(t)}{d} + 4\frac{\lambda_1^2(t)}{d^2}}$, where the formula (39) represents the distance between two objects in a fractal manifold during the first step. We have for all $t \geq 0$,

$$1 \leq a_1(t) < \sqrt{1 + \frac{2\sqrt{3}\delta_1}{d} + 4\frac{\delta_1^2}{d^2}} = a_1, \quad (40)$$

which gives $l_0 \leq l_1(t) < L_1 = l_0 a_1$. The time derivative of the formula (38) corresponds to the instantaneous rate of change of the recession of one ball during the step 1 as measured by an observer on the other ball

$$v_1(t) = \frac{d}{dt} l_1(t) = l_0 \frac{da_1(t)}{dt} = \frac{l_1(t)}{a_1(t)} \frac{da_1(t)}{dt} = l_1(t) \frac{a_1'(t)}{a_1(t)}, \quad (41)$$

and then $v_1(t) \equiv l_1(t)H_1(t)$, where $H_1(t)$ is the Hubble's parameter during the first step given by

$$H_1(t) = \frac{a'_1(t)}{a_1(t)} = \frac{(d\sqrt{3} + 4\lambda_1(t))\lambda'_1(t)}{d^2 a^2(t)}. \quad (42)$$

The signs of the velocity $v_1(t)$ is positive, it is given by the sign of the one step-Hubble's parameter, which is determined by the sign of $\lambda'_1(t)$. Using formula (17) given in Corollary 2, the last result can be generalized to any step n of expansion where the n -th dimensional expanding parameter is given by

$$a_n(t) = \sqrt{1 + \frac{2\sqrt{3}\lambda_n(t)}{d} + 4\frac{\lambda_n^2(t)}{d^2}} < a_n, \quad \forall n \geq 1. \quad (43)$$

5.3 Simultaneous or consecutive expansion

The expansion of an universe defined by a fractal manifold can be described by simultaneous expansion or consecutive expansion. The difficulty is only in modeling the movement of consecutive expansion taking into account the hidden dimensions properties. The consecutive expansion represents a discontinuous expansion, whereas the simultaneous expansion represents a continuous expansion. The continuous expansion seems to be more natural than the discontinuous one, that is why we will focus on it.

5.4 Hubble's law for simultaneous expansion

In simultaneous expansion we have:

During the first step:

$$(1) \begin{cases} l_1(t) = l_0 a_1(t), \\ l_1(t) < L_1 = l_0 a_1, \end{cases} \text{ the maximum distance for the step 1.} \quad (44)$$

During the step 2:

$$(2) \begin{cases} l_2(t) = l_1(t) a_2(t), \\ l_2(t) < L_2 = L_1 a_1, \end{cases} \text{ the maximum distance for the step 2.} \quad (45)$$

During the step n :

$$(n) \begin{cases} l_n(t) = l_{n-1}(t) a_n(t), \\ l_n(t) < L_n = L_{n-1} a_n, \end{cases} \text{ the maximum distance for the step } n. \quad (46)$$

It is not difficult to find by induction over the system (n) that

$$l_n(t) = l_0 \prod_{i=1}^n a_i(t), \quad (47)$$

and then we have the following Hubble's law:

Theorem 6 Let M be a fractal manifold, B_1 and B_2 be two balls distant of $l_0 > 0$. In a simultaneous expansion, the rate of recession of one ball after one step as measured by an observer on the other ball, satisfies the Hubble's law given by

$$v_n(t) \equiv l_n(t)\mathcal{H}_n(t) \quad (48)$$

where $\mathcal{H}_n(t) = \sum_{i=1}^n H_i(t)$ is the n -th partial Hubble's parameter, $H_i(t) = \frac{a'_i(t)}{a_i(t)}$ is the Hubble's parameter during the step i , $l_n(t)$ is the distance between balls for the step n , $(\lambda_n(t))_{n \geq 0}$ are the hidden variables, and $(\delta_n)_{n \geq 0}$ are the hidden dimensions.

Proof: Following the notation (16), if we denote $\mathcal{A}_n(t) = \prod_{i=1}^n a_i(t)$, then formula (47) becomes

$$l_n(t) = l_0 \mathcal{A}_n(t), \quad (49)$$

where the n -th partial expanding parameter is

$$\mathcal{A}_n(t) = \exp \left(\frac{1}{2} \sum_{i=1}^n \ln \left(1 + 2 \frac{\sqrt{3}\lambda_i(t)}{d} + 4 \left(\frac{\lambda_i(t)}{d} \right)^2 \right) \right). \quad (50)$$

The time derivative of the formula (49) corresponds to the rate of recession of one ball during the step n as measured by an observer on the other ball,

$$v_n(t) = \frac{d}{dt} l_n(t) = l_0 \frac{d\mathcal{A}_n(t)}{dt} = \frac{l_n(t)}{\mathcal{A}_n(t)} \frac{d\mathcal{A}_n(t)}{dt} = l_n(t) \frac{\mathcal{A}'_n(t)}{\mathcal{A}_n(t)}. \quad (51)$$

The derivative of the n -th partial expanding parameter $\mathcal{A}_n(t)$ always exists for a finite integer n , and differentiable $\lambda_i(t)$, $i = 1, \dots, n$. Hence we have

$$\frac{d\mathcal{A}_n(t)}{dt} = \mathcal{A}_n(t) \sum_{i=1}^n \frac{(d\sqrt{3} + 4\lambda_i(t))\lambda'_i(t)}{d^2 + 2\lambda_i(t)\sqrt{3}d + 4\lambda_i^2(t)}, \quad (52)$$

with $\lambda'_i(t) = \frac{d\lambda_i(t)}{dt}$. Then the rate of recession of one ball during the step n becomes

$$v_n(t) = l_n(t) \sum_{i=1}^n \frac{(d\sqrt{3} + 4\lambda_i(t))\lambda'_i(t)}{d^2 + 2\lambda_i(t)\sqrt{3}d + 4\lambda_i^2(t)} = l_n(t) \sum_{i=1}^n H_i(t) = l_n(t)\mathcal{H}_n(t), \quad (53)$$

where $\mathcal{H}_n(t)$ is the n -th partial Hubble's parameter.

Remark 6 In the proof of the theorem 6, we find out that for a given finite integer n , $\frac{d\mathcal{A}_n(t)}{dt} = \mathcal{A}_n(t)\mathcal{H}_n(t)$. If the integer n tends to infinity, we need an additional condition to obtain the derivative $\frac{d\mathcal{A}_\infty(t)}{dt}$, where $\mathcal{A}_\infty(t) = \lim_{n \rightarrow +\infty} \mathcal{A}_n(t)$.

Theorem 7 If the sum $\sum_{i=1}^n \lambda'_i(t)$ is uniformly convergent then

$$\frac{d\mathcal{A}_\infty(t)}{dt} = \mathcal{A}_\infty(t) \sum_{i=1}^{+\infty} H_i(t). \quad (54)$$

The main problem in the derivative of the n -th partial sum $\mathcal{A}_n(t)$ as n approaches infinity corresponds to the difficulty to guaranty the uniform convergence of the n -th partial sum. The following lemma guaranties the proof of the theorem.

Lemma 4 1) *The n -th partial sum $\mathcal{A}_n(t)$ is uniformly convergent.*

2) *If the sum $\sum_{i=1}^n \lambda'_i(t)$ is uniformly convergent then the n -th partial Hubble's parameter $\mathcal{H}_n(t)$ is uniformly convergent.*

Proof: 1) It is not difficult to see that $\mathcal{A}_n(t) < A_n$, and because of the convergence of the n -th partial sum A_n we have the normal convergence of $\mathcal{A}_n(t)$, which guaranties the uniformly convergence of $\mathcal{A}_n(t)$.

2) We have

$$\mathcal{H}_n(t) = \sum_{i=1}^n H_i(t) = \sum_{i=1}^n \frac{\left(d\sqrt{3} + 4\lambda_i(t)\right)\lambda'_i(t)}{d^2 + 2\lambda_i(t)\sqrt{3}d + 4\lambda_i^2(t)}.$$

As n approaches infinity, $\lambda_n(t)$ tends to 0, which allows us to write the following equivalence

$$\sum_{i=1}^n \frac{\left(d\sqrt{3} + 4\lambda_i(t)\right)\lambda'_i(t)}{d^2 + 2\lambda_i(t)\sqrt{3}d + 4\lambda_i^2(t)} \approx c \sum_{i=1}^n \lambda'_i(t)$$

and then we find the result.

5.5 The recession velocity for simultaneous expansion

As consequences of the precedent results, we are able to conclude the following:

1) As n approaches infinity, the n -th partial Hubble's parameter \mathcal{H}_n tends to

$$\mathcal{H}_\infty(t) = \frac{\mathcal{A}'_\infty(t)}{\mathcal{A}_\infty(t)} = \sum_{i=1}^{\infty} H_i(t) < \infty. \quad (55)$$

2) In a simultaneous expansion, the distance between balls during the step n is given by $l_n(t) = l_0 \prod_{i=1}^n a_i(t)$, which converges as n approaches $+\infty$ to the distance

$$l_\infty(t) = l_0 \prod_{i=1}^{\infty} a_i(t) < l_0 \prod_{i=1}^{\infty} a_i = L_\infty, \quad (56)$$

where $L_\infty < \infty$ is the maximum distance between balls that can be reached.

3) In a simultaneous expansion, the sequence of recession velocity $v_n(t)$ for a given pair of balls in an homogeneous fractal manifold is increasing and bounded, which guaranties its convergence. Indeed, from the formula (53) we have

$$v_1(t) = l_1(t)H_1(t), \quad (57)$$

$$v_2(t) = l_2(t)\left(H_1(t) + H_2(t)\right), \quad (58)$$

$$v_3(t) = l_3(t) \left(H_1(t) + H_2(t) + H_3(t) \right), \quad (59)$$

$$v_n(t) = l_n(t) \left(H_1(t) + H_2(t) + H_3(t) + \dots + H_n(t) \right), \quad (60)$$

since the Hubble's parameters are positive, and $\forall i \geq 1, l_i(t) \leq l_{i+1}(t)$, then $\forall t \geq 0$

$$v_1(t) < v_2(t) < v_3(t) < \dots < v_n(t). \quad (61)$$

Following (55) and (56) we have

$$v_\infty(t) \equiv \mathcal{H}_\infty(t) l_\infty(t) < \infty, \quad (62)$$

which allows us to assert that in a simultaneous expansion the sequence of recession velocities $v_i(t)$, $\forall i \geq 1$, of a given pair of balls, in an homogeneous fractal manifold, verify $\forall t \geq 0$

$$v_1(t) < v_2(t) < v_3(t) < \dots < v_n(t) < \dots < v_\infty(t) < \infty, \quad (63)$$

meanwhile

$$l_1(t) < l_2(t) < l_3(t) < \dots < l_n(t) < \dots < l_\infty(t) < \infty. \quad (64)$$

If we denote $v_{r_i}(t) = l_i(t) H_i(t)$ the relative recession velocity during the step i , the formula (60) gives the following relation

$$\begin{aligned} v_n(t) &= \left(\prod_{i=2}^n a_i(t) \right) v_{r_1}(t) + \left(\prod_{i=3}^n a_i(t) \right) v_{r_2}(t) + \dots + \left(a_n(t) \right) v_{r_{n-1}}(t) + v_{r_n}(t) \\ v_n(t) &= \sum_{i=1}^{n-1} \left(\prod_{j=i+1}^n a_j(t) \right) v_{r_i}(t) + v_{r_n}(t), \quad \forall n > 1. \end{aligned} \quad (65)$$

Remark 7 1) The order shown in formula (63) doesn't mean that the recession velocity is increasing, it means only that the recession velocities are strictly disjoint, which means that $\forall t \geq 0, \forall i \neq j, v_i(t) \neq v_j(t)$. This result is very important in analysis and interpretations of observation data of galaxies, indeed, the measure of recession velocity of galaxies by "Cosmological Redshift" method for different period of time might gives an increasing recession velocity that represents in reality an increasing value of partial sum of velocities, whereas the instantaneous rate of change of the recession distance between galaxies could be negative. The increasing values of the recession velocities data between well separated galaxies can not be interpreted as an acceleration of the expansion of the universe if we don't know the nature of the expansion (consecutive or simultaneous or any other form).

2) The formula (63) does not represent the velocity of the expansion of the universe, it is only a recession partial sum of relative velocities between separated balls during the step from 1 to n (see(65)).

3) The formula (62) means that as n tends to ∞ , the recession velocity is independent of dimension, which means that there is no variation of geometry (no apparition of new structures) and then no more expansion.

5.6 The nature of the expansion

From the formulas (63), and (64) it is impossible to confirm that we are in presence of an accelerating or decelerating expansion, we know only that this expansion will stop as n tends to the infinity. If so, then there should exist some deceleration somewhere to explain how the recession of balls will stop. We introduce the growth velocity to evaluate the recession distance between balls after two successive steps that will clarify the real nature of the expansion, and this will be valid for consecutive expansion or simultaneous expansion.

Definition 14 *Let M be a fractal manifold, A and B two distant and fixed balls on it. Let L_n be the distance between A and B after n steps. We define the growth velocity of distance between balls A and B and we denoted V_{δ_n} the quantity:*

$$V_{\delta_n} = L_{n+1} - L_n = l_0 A_n \left(\sqrt{1 + 2\sqrt{3}\frac{\delta_{n+1}}{d} + 4\left(\frac{\delta_{n+1}}{d}\right)^2} - 1 \right). \quad (66)$$

Theorem 8 *Let M be a fractal manifold, A and B two distant and fixed balls on it. Then there exist $n_0 \in \mathbb{N}$ such that $\forall n > n_0$, the growth velocity of distance between balls A and B is decreasing.*

Proof: To prove that V_{δ_n} is decreasing as n tends to infinity, it is sufficient to look after $\frac{V_{\delta_{n+1}}}{V_{\delta_n}}$. We have $\frac{V_{\delta_{n+1}}}{V_{\delta_n}} = a_{n+1} \left(\frac{a_{n+2}-1}{a_{n+1}-1} \right)$. To obtain the result, it is sufficient to prove

$$a_{n+1} \left(\frac{a_{n+2}-1}{a_{n+1}-1} \right) \leq 1. \quad (67)$$

Indeed, we have

$$\left(1 + 2\sqrt{3}\frac{\delta_{n+1}}{d} + 4\left(\frac{\delta_{n+1}}{d}\right)^2 \right)^{-1} \approx_{+\infty} 1 - \left(2\sqrt{3}\frac{\delta_{n+1}}{d} + 4\left(\frac{\delta_{n+1}}{d}\right)^2 \right) + o(\delta_{n+1}), \quad (68)$$

$$\begin{aligned} \text{then } & 1 + 2\sqrt{3}\frac{\delta_{n+2}}{d} + 4\left(\frac{\delta_{n+2}}{d}\right)^2 + \left(1 + 2\sqrt{3}\frac{\delta_{n+1}}{d} + 4\left(\frac{\delta_{n+1}}{d}\right)^2 \right)^{-1} + 2\frac{a_{n+2}}{a_{n+1}} \\ & \approx_{+\infty} 1 + 2\sqrt{3}\frac{\delta_{n+2}}{d} + 4\left(\frac{\delta_{n+2}}{d}\right)^2 + 1 - \left(2\sqrt{3}\frac{\delta_{n+1}}{d} + 4\left(\frac{\delta_{n+1}}{d}\right)^2 \right) + 2\frac{a_{n+2}}{a_{n+1}} + o(\delta_{n+1}) \\ & = 2 + \left(2\sqrt{3}\frac{\delta_{n+2}}{d} + 4\left(\frac{\delta_{n+2}}{d}\right)^2 \right) - \left(2\sqrt{3}\frac{\delta_{n+1}}{d} + 4\left(\frac{\delta_{n+1}}{d}\right)^2 \right) + 2\frac{a_{n+2}}{a_{n+1}} + o(\delta_{n+1}), \end{aligned}$$

then there exists $n_0 \in \mathbb{N}$, such that $\forall n > n_0$ we have:

$$1 + 2\sqrt{3}\frac{\delta_{n+2}}{d} + 4\left(\frac{\delta_{n+2}}{d}\right)^2 + \left(1 + 2\sqrt{3}\frac{\delta_{n+1}}{d} + 4\left(\frac{\delta_{n+1}}{d}\right)^2 \right)^{-1} + 2\frac{a_{n+2}}{a_{n+1}} \leq 2 + 2\frac{a_{n+2}}{a_{n+1}} \leq 4$$

$$\text{which gives } \sqrt{1 + 2\sqrt{3}\frac{\delta_{n+2}}{d} + 4\left(\frac{\delta_{n+2}}{d}\right)^2} + \left(\sqrt{1 + 2\sqrt{3}\frac{\delta_{n+1}}{d} + 4\left(\frac{\delta_{n+1}}{d}\right)^2} \right)^{-1} \leq 2,$$

then $a_{n+2} + \frac{1}{a_{n+1}} \leq 2$, which gives $a_{n+2} - 1 \leq 1 - \frac{1}{a_{n+1}}$, then $\frac{a_{n+2}-1}{a_{n+1}-1} \leq \frac{1}{a_{n+1}}$, to obtain $a_{n+1} \frac{a_{n+2}-1}{a_{n+1}-1} \leq 1$, $\forall n > n_0$, which conclude the proof.

Remark 8 *The last theorem confirms the deceleration of the expansion in spite of the increasing sequence of the recession velocity.*

6 Global Impact Toward New Principles

The previous theoretical study used sound arguments to demonstrate how the expansion of a universe, defined by a fractal manifold, works. A reciprocal causality between variation of geometry and matter has been deduced, which allows us to state the following findings, bearing in mind that their analysis is not exhaustive:

6.1 Geometrical findings

The physical universe can be represented by a fractal manifold where the variable metric at the step n that defines the distance of the space-time event is given in reduced form (in linear coordinate) by:

$$d\tau_n^2 = c^2 dt^2 - \left(\prod_{i=1}^n a_i^2(t) \right) ds^2, \quad (69)$$

where ds^2 is the Newton spatial distance. The automatic formation of new structures in fractal manifold which is subject to steps gives the following principles:

- *The universe has geometric properties which are independent of matter that it contains.*
- *The variation of the universe geometry creates the movement of matter.* Galaxies become distant from each other (appear going away from each other), because of their own constant dimensions with respect to the increasing dimension of the universe.
- *The variation of the universe geometry affects the gravity.* A natural consequence of the expansion.
- *The variation of the universe geometry affects the time.* Indeed, it affects the celestial movement and stretches the light wavelength along with the universe. The time will be affected by dilation or contraction following the nature of the geometrical variation.
- *In an expanding space where points are expanding, there is no straight lines geodesic. All geodesics are curved due to the expansion of points.* Since $\prod_{i=1}^n a_i^2(t) \neq 1$ in (69), then the space time defined by a fractal manifold is a curved space time.

6.2 Physical consequences

To use fractal manifold in the description of our universe, we have to consider a galaxy as a whole solid and then postulate the following: *The gravitational interaction of matter generates an interaction effect of a deformable system with external objects, equivalent to the interaction of a solid system which has a mass⁸, a variable inertial center, and an inertial reaction in accelerating movement.* Using this postulate, we deduce from the previous study the following:

- *There must exist another gravity created by the deceleration of the expansion of the universe, and this gravity has the same direction as the recession velocity of galaxies.*

⁸The distribution of mass is not uniform.

- *There exists one region in each galaxy where this gravity force is huge and occupies a location. This locus represents the inertial center of each galaxy.* Following the non homogeneous repartition of matter in galaxies, there exist some regions where the inertial force is more or less intense following the density of matter. If we identify these regions as black holes, then black holes are a natural consequence of the deceleration of the expansion of the universe⁹. The locus of the black holes is not fixed in the universe (since planets and stars are moving, and many stars may disappear by explosion), and their movement is not a movement of planets or stars, it's a movement of a variable inertial center. It may appear in some region and disappear after the death of some stars, and appear in another region that represents the new inertial center of the system. This gravity looks like a supergiant vacuum cleaner that sucks in everything insight, it will suck only dust and everything left after the explosion of stars or anything in free movement. This huge gravity can be evaluated if we can approximate the total mass of galaxy. This region is defined by its position (inertial center) and its inertial force (mass of galaxy).

- *This huge gravity will vanish when the expansion of the universe stops, and it will re-appear when the universe is in an accelerating contraction state. Its direction is opposite to the direction of the universe contraction.* In a constant expansion of the universe, there is neither inertial force nor black holes.

- *The variation of the universe geometry bends the light.* A new characteristic of a curved path (sinusoidal path with a countable number of non differentiability points) of light in the universe might explain the wave appearance of the light, and it could bring up and resolve the problem of duality wave-corpuscle of the light. More analysis and detail about the light in an expanding universe will be announced later on [2]. Finally, using fractal manifold, here is the most probable scenario of its fate: *Our world began with the Big Bang in which the Universe was very hot and extremely dense. This Big Bang put our universe in an accelerating expansion causing the temperature to drop and matter/energy to spread out. The acceleration of the expansion reached its maximum and started to decelerate after the formation of planets, stars and galaxies (under the gravitational attraction of matter). Following this deceleration, black holes will appear in the inertial center of each galaxy and each inertial center of matter distribution. The universe continues its decelerating expansion under the effect of kinetic energy causing time dilation. This expansion will stop one day, and the black holes will then vanish. The mass creates gravity, which will pull on everything and leads the universe to start contracting. The black holes will re-emerge due to acceleration, and we will have time contraction. The pulling must lead everything to collapse in a "Big Crunch". Some planets will collapse before others (the nearest first). In the "Big Crunch", the universe will be very hot and extremely dense. The temperature will attain an extremely high level and any gas at a temperature exceeding zero kelvin is bound to expand to wherever space is available to it. The universe will expand again causing the temperature to drop and matter/energy to spread out, which will give new structures that will be different from what we observe today; a new sky, new planets and maybe a new Earth.*

⁹This is more rational than the collapse of matter to the point of zero volume and infinite density.

A Expanding Fractal Manifold

In order to prove that all fractal manifolds are expanding, we explain how objects in fractal manifolds are expanding following the different steps of the diagram Fig.1.

Let us consider an object $P = Rg(x)$ of a fractal manifold M , where x is an internal structure on it. Let $(\Omega, \varphi_1, \varphi_2)$ be a local chart at the object P of M , where $\Omega = \bigcup_{\delta_0 \in \mathcal{R}_f} \Omega_{\delta_0}$ is an open set of M , $\varphi_1 = (\varphi_{\delta_0})_{\delta_0}$ is a family of homeomorphisms φ_{δ_0} from Ω_{δ_0} to an open set $V_{\delta_0}^+$ of $\prod_{i=1}^3 \Gamma_{i\delta_0}^+ \times \{\delta_0\}$, $\forall \delta_0 \in \mathcal{R}_f$, and $\varphi_2 = (T_{\delta_0} \circ \varphi_{\delta_0})_{\delta_0}$ is a family of homeomorphisms $T_{\delta_0} \circ \varphi_{\delta_0}$ from Ω_{δ_0} to an open set $V_{\delta_0}^-$ of $\prod_{i=1}^3 \Gamma_{i\delta_0}^- \times \{\delta_0\}$ for all $\delta_0 \in \mathcal{R}_f$ (see Fig.1A).

$$\begin{array}{ccc}
 M = \bigcup_{\delta_0 \in \mathcal{R}_f} M_{\delta_0} & \xrightarrow{\varphi_1 = (\varphi_{\delta_0})_{\delta_0}} & \bigcup_{\delta_0 \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\delta_0}^+ \times \{\delta_0\} \\
 & \searrow \varphi_2 = (T_{\delta_0} \circ \varphi_{\delta_0})_{\delta_0} & \downarrow (T_{\delta_0})_{\delta_0} \\
 & & \bigcup_{\delta_0 \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\delta_0}^- \times \{\delta_0\}
 \end{array}$$

Fig.1A. - Diagram of fractal manifold that represents the first step.

Using the definition 6, this object is represented in the triplet local chart $(\Omega, \varphi_1, \varphi_2)$ by two strings:

$$Rg(x^+) \cup Rg(x^-) = \left(\bigcup_{\delta_0 \in \mathcal{R}_f} \varphi_{\delta_0}(x(\delta_0)) \right) \cup \left(\bigcup_{\delta_0 \in \mathcal{R}_f} T_{\delta_0} \circ \varphi_{\delta_0}(x(\delta_0)) \right) \quad (70)$$

where x^+ and x^- are given by formula (3) by substituting δ_0 for ε . More precisely:

- 1) For all $x^+(\delta_0) \in Rg(x^+)$, there exists $(x_1, y_1, x_2, y_2, x_3, y_3) \in \mathbb{R}^6$ such that

$$x^+(\delta_0) = \varphi_{\delta_0}(x(\delta_0)) = (x_1, y_1, x_2, y_2, x_3, y_3), \quad \text{with } x_i = x_i(\delta_0), \quad \text{and}$$

$$y_i = y_i^+(x_i, \delta_0) = \frac{1}{\delta_0} \int_{x_i}^{x_i + \delta_0} f(s) ds, \quad (71)$$

$$\forall \delta_0 \in \mathcal{R}_f, i = 1, 2, 3.$$

- 2) For all $x^-(\delta_0) \in Rg(x^-)$, there exists $(x_1, y_1, x_2, y_2, x_3, y_3) \in \mathbb{R}^6$ such that

$$x^-(\delta_0) = T_{\delta_0} \circ \varphi_{\delta_0}(x(\delta_0)) = (x_1, y_1, x_2, y_2, x_3, y_3), \quad \text{with } x_i = x_i(\delta_0), \quad \text{and}$$

$$y_i = y_i^-(x_i, \delta_0) = \frac{1}{\delta_0} \int_{x_i - \delta_0}^{x_i} f(s) ds, \quad (72)$$

$\forall \delta_0 \in \mathcal{R}_f, i = 1, 2, 3.$

By corollary 1, there exists a quintuplet local chart $(\Omega, \varphi_3 \circ \varphi_1, \varphi_4 \circ \varphi_1, \varphi_5 \circ \varphi_2, \varphi_6 \circ \varphi_2)$ at the object P , where $\varphi_3, \varphi_4, \varphi_5, \varphi_6$, are families of homeomorphisms given by:

$$\begin{array}{ccc}
 \bigcup_{\delta_0 \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\delta_0}^+ \times \{\delta_0\} & \xrightarrow{\varphi_3 = ((\varphi_{\delta_1})_{\delta_1, \delta_0}^+)^{\delta_0}} & \bigcup_{\delta_0 \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_1}^{+,+} \times \{\delta_1\} \times \{\delta_0\} \\
 & \searrow \varphi_4 = ((T_{\delta_1} \circ \varphi_{\delta_1})_{\delta_1, \delta_0}^+)^{\delta_0} & \downarrow T_2 = ((T_{\delta_1})_{\delta_1, \delta_0}^+)^{\delta_0} \\
 & & \bigcup_{\delta_0 \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_1}^{+,-} \times \{\delta_1\} \times \{\delta_0\}
 \end{array}$$

$$\begin{array}{ccc}
 \bigcup_{\delta_0 \in \mathcal{R}_f} \prod_{i=1}^3 \Gamma_{i\delta_0}^- \times \{\delta_0\} & \xrightarrow{\varphi_5 = ((\varphi_{\delta_1})_{\delta_1, \delta_0}^-)^{\delta_0}} & \bigcup_{\delta_0 \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_1}^{-,+} \times \{\delta_1\} \times \{\delta_0\} \\
 & \searrow \varphi_6 = ((T_{\delta_1} \circ \varphi_{\delta_1})_{\delta_1, \delta_0}^-)^{\delta_0} & \downarrow T_3 = ((T_{\delta_1})_{\delta_1, \delta_0}^-)^{\delta_0} \\
 & & \bigcup_{\delta_0 \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_1}^{-,-} \times \{\delta_1\} \times \{\delta_0\}
 \end{array}$$

where for $\sigma_1 = \pm$, $(\varphi_{\delta_1})_{\delta_1, \delta_0}^{\sigma_1}$ represents the family $(\varphi_{\delta_1})_{\delta_1 \in \mathcal{R}_{\delta_1}}^{\sigma_1}$ at the resolution δ_0 (respectively for $(T_{\delta_1})_{\delta_1, \delta_0}^{\sigma_1}$ and $(T_{\delta_1} \circ \varphi_{\delta_1})_{\delta_1, \delta_0}^{\sigma_1}$), with $\mathcal{R}_{\delta_1} = [0, \lambda]$, $0 < \lambda \ll 1$ a small real number. Then we have two new diagrams for the fractal manifold M given by:

$$\begin{array}{ccc}
 M & \xrightarrow{\varphi_3 \circ \varphi_1} & \bigcup_{\delta_0 \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_1}^{+,+} \times \{\delta_1\} \times \{\delta_0\} \\
 & \searrow \varphi_4 \circ \varphi_1 & \downarrow T_2 \\
 & & \bigcup_{\delta_0 \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_1}^{+,-} \times \{\delta_1\} \times \{\delta_0\}.
 \end{array}$$

$$\begin{array}{ccc}
 M & \xrightarrow{\varphi_5 \circ \varphi_2} & \bigcup_{\delta_0 \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_1}^{-,+} \times \{\delta_1\} \times \{\delta_0\} \\
 & \searrow \varphi_6 \circ \varphi_2 & \downarrow T_3 \\
 & & \bigcup_{\delta_0 \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_1}^{-,-} \times \{\delta_1\} \times \{\delta_0\}.
 \end{array}$$

Following these two diagrams, the same point P is represented in the quintuplet local chart: $(\Omega, \varphi_3 \circ \varphi_1, \varphi_4 \circ \varphi_1, \varphi_5 \circ \varphi_2, \varphi_6 \circ \varphi_2)$ by two surfaces given by

$Rg(x^{++}) \cup Rg(x^{+-}) \cup Rg(x^{-+}) \cup Rg(x^{--})$, with

$$Rg(x^{++}) = \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \bigcup_{\delta_0 \in \mathcal{R}_f} \varphi_3 \circ \varphi_1(x(\delta_0)), \quad Rg(x^{+-}) = \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \bigcup_{\delta_0 \in \mathcal{R}_f} \varphi_4 \circ \varphi_1(x(\delta_0)),$$

$$Rg(x^{-+}) = \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \bigcup_{\delta_0 \in \mathcal{R}_f} \varphi_5 \circ \varphi_2(x(\delta_0)), \quad Rg(x^{--}) = \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \bigcup_{\delta_0 \in \mathcal{R}_f} \varphi_6 \circ \varphi_2(x(\delta_0)),$$

and where

$$\begin{aligned} x^{++} : \mathcal{R}_f \times \mathcal{R}_{\delta_1} &\longrightarrow \bigcup_{\delta_0 \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_0}^{++} \times \{\delta_1\} \times \{\delta_0\} \\ (\delta_0, \delta_1) &\longmapsto (\varphi_{\delta_1})_{\delta_1, \delta_0}^+ \circ \varphi_{\delta_0}(x(\delta_0)), \end{aligned} \quad (73)$$

$$\begin{aligned} x^{+-} : \mathcal{R}_f \times \mathcal{R}_{\delta_1} &\longrightarrow \bigcup_{\delta_0 \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_0}^{+-} \times \{\delta_1\} \times \{\delta_0\} \\ (\delta_0, \delta_1) &\longmapsto (T_{\delta_1} \circ \varphi_{\delta_1})_{\delta_1, \delta_0}^+ \circ \varphi_{\delta_0}(x(\delta_0)), \end{aligned} \quad (74)$$

$$\begin{aligned} x^{-+} : \mathcal{R}_f \times \mathcal{R}_{\delta_1} &\longrightarrow \bigcup_{\delta_0 \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_0}^{-+} \times \{\delta_1\} \times \{\delta_0\} \\ (\delta_0, \delta_1) &\longmapsto (\varphi_{\delta_1})_{\delta_1, \delta_0}^- \circ (T_{\delta_0} \circ \varphi_{\delta_0})(x(\delta_0)), \end{aligned} \quad (75)$$

$$\begin{aligned} x^{--} : \mathcal{R}_f \times \mathcal{R}_{\delta_1} &\longrightarrow \bigcup_{\delta_0 \in \mathcal{R}_f} \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \prod_{i=1}^3 \Gamma_{i\delta_0}^{--} \times \{\delta_1\} \times \{\delta_0\} \\ (\delta_0, \delta_1) &\longmapsto (T_{\delta_1} \circ \varphi_{\delta_1})_{\delta_1, \delta_0}^- \circ (T_{\delta_0} \circ \varphi_{\delta_0})(x(\delta_0)), \end{aligned} \quad (76)$$

more precisely, we have

1) For $Rg(x^{++})$, there exists $(x_1, y_1, x_2, y_2, x_3, y_3) \in \mathbb{R}^6$ such that $(\varphi_{\delta_1})_{\delta_1, \delta_0}^+ \circ \varphi_{\delta_0}(x(\delta_0)) = (x_1, y_1, x_2, y_2, x_3, y_3)$, with $x_i = x_i(\delta_0, \delta_1)$, and

$$y_i = y_i^{++}(x_i, \delta_0, \delta_1) = \frac{1}{\delta_0 \delta_1} \int_{x_i}^{x_i + \delta_1} \int_t^{t + \delta_0} f(s) ds dt, \quad (77)$$

$\forall (\delta_0, \delta_1) \in \mathcal{R}_f \times \mathcal{R}_{\delta_1}, i = 1, 2, 3.$

2) For $Rg(x^{+-})$, there exists $(x_1, y_1, x_2, y_2, x_3, y_3) \in \mathbb{R}^6$ such that $(T_{\delta_1} \circ \varphi_{\delta_1})_{\delta_1, \delta_0}^+ \circ \varphi_{\delta_0}(x(\delta_0)) = (x_1, y_1, x_2, y_2, x_3, y_3)$, with $x_i = x_i(\delta_0, \delta_1)$, and

$$y_i = y_i^{+-}(x_i, \delta_0, \delta_1) = \frac{1}{\delta_0 \delta_1} \int_{x_i}^{x_i + \delta_1} \int_{t - \delta_0}^t f(s) ds dt, \quad (78)$$

$\forall (\delta_0, \delta_1) \in \mathcal{R}_f \times \mathcal{R}_{\delta_1}, i = 1, 2, 3.$

3) For $Rg(x^{-+})$, there exists $(x_1, y_1, x_2, y_2, x_3, y_3) \in \mathbb{R}^6$ such that

$(\varphi_{\delta_1})_{\delta_1, \delta_0}^- \circ (T_{\delta_0} \circ \varphi_{\delta_0})(x(\delta_0)) = (x_1, y_1, x_2, y_2, x_3, y_3)$, with $x_i = x_i(\delta_0, \delta_1)$, and

$$y_i = y_i^{-+}(x_i, \delta_0, \delta_1) = \frac{1}{\delta_0 \delta_1} \int_{x_i - \delta_1}^{x_i} \int_t^{t + \delta_0} f(s) ds dt, \quad (79)$$

$\forall (\delta_0, \delta_1) \in \mathcal{R}_f \times \mathcal{R}_{\delta_1}, i = 1, 2, 3$.

4) For $Rg(x^{--})$, there exists $(x_1, y_1, x_2, y_2, x_3, y_3) \in \mathbb{R}^6$ such that $(T_{\delta_1} \circ \varphi_{\delta_1})_{\delta_1, \delta_0}^- \circ (T_{\delta_0} \circ \varphi_{\delta_0})(x(\delta_0)) = (x_1, y_1, x_2, y_2, x_3, y_3)$, with $x_i = x_i(\delta_0, \delta_1)$, and

$$y_i = y_i^{--}(x_i, \delta_0, \delta_1) = \frac{1}{\delta_0 \delta_1} \int_{x_i - \delta_1}^{x_i} \int_{t - \delta_0}^t f(s) ds dt, \quad (80)$$

$\forall (\delta_0, \delta_1) \in \mathcal{R}_f \times \mathcal{R}_{\delta_1}, i = 1, 2, 3$.

With the previous notations of $y^{\sigma_1 \sigma_2}$ and y^σ for $\sigma_1 = \pm, \sigma_2 = \pm, \sigma = \pm$, we have the following results:

Theorem 9 *The local representation of object in fractal manifold verifies:*

$$\left(Rg(x^+) \cup Rg(x^-) \right) \subset \left(Rg(x^{++}) \cup Rg(x^{+-}) \cup Rg(x^{-+}) \cup Rg(x^{--}) \right) \quad (81)$$

Proof: Using lemma 2, we have

$$\begin{aligned} & Rg(x^{++}) \cup Rg(x^{+-}) \cup Rg(x^{-+}) \cup Rg(x^{--}) = \\ & \left(Rg(x^+) \cup Rg(x^-) \right) \cup \left(Rg(x^{++})^* \cup Rg(x^{+-})^* \cup Rg(x^{-+})^* \cup Rg(x^{--})^* \right), \end{aligned}$$

where

$$\begin{aligned} Rg(x^{++})^* &= \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \bigcup_{\delta_0 \in \mathcal{R}_f}^{\delta_1 \neq 0} \varphi_3 \circ \varphi_1(x_{\delta_0}(\delta_1)), \\ Rg(x^{+-})^* &= \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \bigcup_{\delta_0 \in \mathcal{R}_f}^{\delta_1 \neq 0} \varphi_4 \circ \varphi_1(x_{\delta_0}(\delta_1)), \\ Rg(x^{-+})^* &= \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \bigcup_{\delta_0 \in \mathcal{R}_f}^{\delta_1 \neq 0} \varphi_5 \circ \varphi_2(x_{\delta_0}(\delta_1)), \\ Rg(x^{--})^* &= \bigcup_{\delta_1 \in \mathcal{R}_{\delta_1}} \bigcup_{\delta_0 \in \mathcal{R}_f}^{\delta_1 \neq 0} \varphi_6 \circ \varphi_2(x_{\delta_0}(\delta_1)), \end{aligned}$$

which gives the result.

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